ON THE CONCEPTS OF SIMPLE LOADING AND ON POSSIBLE DEFORMATION PATHS

(O PONIATIIAKH PROSTOGO NAGRUZHENIIA I O VOZMOZHNYKH PUTIAKH DEFORMATSII)

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In the literature on the theory of plasticity one can encounter discussions of simple loading, by which is meant an equilibrium process of a proportional change of all components of the stress tensor and correspondingly all components of the strain tensor for all particles of the body. The determination of the simple loading is based on the assumption that the proportionality coefficient for the increase of the components of the stress tensor and the coefficient k for the increase of the components of the strain tensor depend only on time. Below we will not deal with the not-uninteresting question of the character of the particular ideal physical properties which the deforming body should possess to permit a process of simple loading.

We will study the phenomenon of the deformation of a continuous medium from the geometrical point of view, apart from the physical nature of that medium. The derivations following below are applicable to gaseous, liquid and plastic or solid bodies.

We will demonstrate that with an exact analysis of the phenomenon of the deformation of any continuous medium, the process of deformation, in which the components of the strain tensor vary according to the proportional law given above, may correspond only to deformations of some special type for the entire body as a whole. Of course, a state of strain of a particular kind can be assured for every body only by a very special type of external surface loading.

It follows from this that apart from the physical nature of the body the concept of simple loading for arbitrarily distributed external surface forces has, in general, no meaning for finite deformations.

To prove the statement formulated above, we will take some system of coordinates ξ^1, ξ^2, ξ^3 fixed in the medium, and analyze two positions of the body: first, the original position, in which an element of length is given by the formula

$$ds_0^2 = g_{\alpha s}{}^{\circ}\, d\xi^{\alpha}\, d\xi^{\beta} \tag{1}$$

and second, the final position, in which the element ds_0 corresponds to the element ds, where

$$ds^2 = \hat{g}_{\alpha\beta}\, d\xi^{\alpha}\, d\xi^{\beta} \tag{2}$$

In these and the following formulas, summation is assumed to take place on repeated upper and lower indices.

As is well known, the covariant components of the strain tensor are given by the formula

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\hat{g}_{\alpha\beta} - g_{\alpha\beta}{}^{\circ}) \tag{3}$$

Let us assume that the deformed state under investigation can be obtained from the original state by means of a simple loading where the strain tensors for the intermediate states are given by the formulas

$$\varepsilon_{\alpha\beta}{}^{*} = k\,(t)\,\varepsilon_{\alpha\beta} = \frac{1}{2}(g_{\alpha\beta}{}^{*} - g_{\alpha\beta}{}^{\circ}) \qquad (\alpha, \beta = 1, 2, 3) \tag{4}$$

where $0 \leqslant k \leqslant 1$, $k(0) = 0$ corresponds to the first original state, and $k(\hat{t}) = 1$ corresponds to the second state.

Let us now recall the nature and the derivation of the geometrical compatibility conditions for the components of the strain tensor. Since the quadratic forms of the squares of the elements of length determine the element of length in the Euclidean space, the Riemann tensor goes to zero for the fundamental tensor $g_{\alpha\beta}{}^{*}$. This gives the relation

$$R_{ij\mu\nu} = \frac{\partial \Gamma_{\nu\mu i}}{\partial \xi^{j}} - \frac{\partial \Gamma_{\nu\mu j}}{\partial \xi^{i}} + g^{*\alpha\omega}\,[\Gamma_{\omega\mu j}\,\Gamma_{\alpha\nu i} - \Gamma_{\omega\mu i}\,\Gamma_{\alpha\nu j}] = 0 \tag{5}$$

where

$$\Gamma_{\nu\alpha j} = \frac{1}{2}\left[\frac{\partial g_{\nu\alpha}{}^{*}}{\partial \xi^{j}} + \frac{\partial g_{j\nu}{}^{*}}{\partial \xi^{\alpha}} - \frac{\partial g_{\alpha j}{}^{*}}{\partial \xi^{\nu}}\right] \tag{6}$$

and the matrix $\| g^{*\alpha\beta} \|$ is the inverse of $\| g_{\alpha\beta}{}^{*} \|$.

In the general case equation (5) represents a system of nonlinear second-order partial differential equations consisting of six independent equations corresponding to the following systems of indices:

$$ij\,\mu\nu = 1212,\ 1313,\ 2323,\ 1213,\ 2123,\ 3132 \tag{7}$$

The system of coordinates ξ^1, ξ^2, ξ^3 fixed in the medium is arbitrary and deforms together with the material. This coordinate system may always be assumed to be in the original position a Cartasian one, and consequently the components of the tensor ${g_{\alpha\beta}}^0$ do not depend on ξ^1, ξ^2, ξ^3. When this fact is utilized, the compatibility conditions for the components of the tensor $\epsilon_{\alpha\beta}$ can be written on the basis of (4) and (5) in the form

$$\frac{\partial G_{\nu\mu i}}{\partial \xi^j} - \frac{\partial G_{\nu\mu j}}{\partial \xi^i} + k g^{*\alpha\omega} [G_{\omega\mu j} G_{\alpha\nu i} - G_{\omega\mu i} G_{\alpha\nu j}] = 0 \tag{8}$$

where

$$G_{\nu\alpha j} = \frac{\partial \varepsilon_{\alpha\nu}}{\partial \xi^j} + \frac{\partial \varepsilon_{j\nu}}{\partial \xi^\alpha} - \frac{\partial \varepsilon_{\alpha j}}{\partial \xi^\nu} \tag{9}$$

and where in the base ${g_{\alpha\beta}}^*$ the contravariant components of $\| g^{*\alpha\beta} \|$ are determined from the matrix equation

$$\| g^{*\alpha\omega} \| = \| {g_{\alpha\beta}}^\circ + 2k\, \varepsilon_{\alpha\beta} \|^{-1} \tag{10}$$

With $k \to 0$ we have $g^{*\alpha\beta} \to g^{0\alpha\beta}$.

The *coefficient* k can assume arbitrary values in the interval (0.1), and thus the system of equations (8) separates into the following two systems of equations

$$\frac{\partial G_{\nu\mu i}}{\partial \xi^j} - \frac{\partial G_{\nu\mu j}}{\partial \xi^i} = 0 \tag{11}$$

$$g^{*\alpha\omega} [G_{\omega\mu j} G_{\alpha\nu i} - G_{\omega\mu i} G_{\alpha\nu j}] = 0 \tag{12}$$

where the components of $g^{*\alpha\omega}$ depend on the arbitrary value of k, which leads to an increased number of equations on $\epsilon_{\alpha\beta}$ in system (12).

The system of equations (11) for the components of the finite strain tensor $\epsilon_{\alpha\beta}$ coincides with the system of St. Venant equations for the components of the infinitesimally small strain. The general solution of system (11) can be presented in the following form:

$$G_{\nu\mu i} = G_{\nu i \mu} = \frac{\partial^2 \Omega_\nu}{\partial \xi^i \partial \xi^\mu}, \quad \varepsilon_{\alpha\beta} = \frac{1}{2} \left(\frac{\partial \Omega_\alpha}{\partial \xi^\beta} + \frac{\partial \Omega_\beta}{\partial \xi^\alpha} \right) \tag{13}$$

where $\Omega_1(\xi^1, \xi^2, \xi^3)$, $\Omega_2(\xi^1, \xi^2, \xi^3)$ and $\Omega_3(\xi^1, \xi^2, \xi^3)$ are three arbitrary functions of their own arguments. In the approximate linear theory the quantities Ω_1, Ω_2, Ω_3 can be considered as vector components of displacements of a point of the medium.

Equations (12) can be considered as additional equations limiting the form of the functions $\Omega_k(\xi^1, \xi^2, \xi^3)$ and correspondingly the form of the functions $\epsilon_{\alpha\beta}(\xi^1, \xi^2, \xi^3)$, giving the strain which allows a simple loading.

Obviously, equations (11) to (12) are satisfied if the strain is homogeneous (affine), when $\epsilon_{\alpha\beta}$ = const (α, β = 1, 2, 3).

As an example we will study a strain which for an orthogonal Cartesian coordinate system is given by the following formulas:

$$\Omega_1 = \xi^3, \ \Omega_2 = h(\xi^3)\,\xi^1, \ \Omega_3 = 0$$

and correspondingly

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 0, \ \varepsilon_{12} = \frac{1}{2} h(\xi^3), \ \varepsilon_{13} = \frac{1}{2}, \ \varepsilon_{23} = \frac{1}{2} h'(\xi^3)\,\xi^1 \qquad (14)$$

where $h(\xi^3)$ is an arbitrary function which is not constant.

It can be easily verified that with $k = 1$ the system of functions (14) satisfies all equations of system (12) and with $k \neq 1$ one of the equations (12) corresponding to the indices $i = 1$, $j = 3$, $\mu = 1$, $\nu = 3$ is not satisfied. Consequently, the components of the tensor $\epsilon_{\alpha\beta}^* = k\epsilon_{\alpha\beta}$, where $\epsilon_{\alpha\beta}$, given by formula (14), do not satisfy the compatibility conditions.

Thus the geometrically realized deformation given by tensor (14) cannot be obtained from the initial state by means of the process of simple loading, for geometrical reasons.

If the deformations are infinitesimally small and only small terms are preserved in the compatibility conditions, then the compatibility conditions are satisfied, with a proportional change of the components of the strain tensor at every point of the body for any strain distribution allowable by the compatibility conditions.

Thus, because of its nature, the concept of simple loading can be only applied within the framework of the approximate linear theory, which is valid only for geometrically small deformations.

It is well known that in the typical and basic problems of the theory of motion of a viscous liquid, and in the theory of an ideally plastic medium, the deformations are finite, and thus the extension of the motion of simple loading to this case should not supposed to be always possible.

Of course, in the general case, the strain distribution inside the body imposes some limitations of a geometrical character upon the possible deformation paths for neighboring particles of the medium when the external conditions prescribe some process of deformation of the body as a whole.

Translated by M.I.Y.